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Anti-exceedances in Permutation Groups

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Given an arbitrary positive integer d, we investigate the hypotheses under which the elements of a permutation group containing the cycle $(1\,2\cdots n)$ that are not powers of this cycle have more than d anti-exceedances. We also use the combinatorial properties of anti-exceedances to prove some algebraic facts about Frobenius groups.

1. Introduction

A permutation π of the symmetric group presents an exceedance in i whenever $i\pi > i$. Otherwise we say that π presents an anti-exceedance. The notion of exceedance of a permutation is one which often occurs in combinatorics; statistics on exceedances are the object of several theorems and works (see [3], [4] or [6]).

The starting point of this work is a result of D. Perrin [8] and [9]. This result relates the number of anti-exceedances to the minimal degree of the permutation group, and it is used to build finite codes associated with a given permutation group.

Our main result (Theorem 1) gives a lower bound for the degree n in groups containing the cycle $\alpha = (1\,2\cdots n)$. If n is greater than this bound, then every permutation outside the cyclic subgroup $\langle\alpha\rangle$ has a number of anti-exceedances greater than a previously fixed positive integer d; this bound is a function of the number d and of the minimal degree $n-k$ of the group.

Our result generalizes Lemma 2 of [9], since the number d is here independent of k and, in the special case considered in [9], it gives a lower value for the bound. We also prove that this bound is, in a certain sense, the best possible.

The second part of the paper deals in particular with Frobenius groups; that is, groups of degree n and minimal degree $n-1$. With this further hypothesis, Theorem 1 takes a more precise form; anti-exceedances are also used in this part as combinatorial tools to prove some algebraic properties of Frobenius groups, such as conditions about the parity of its degree or the order of its kernel and complement.

2. Basic Definitions and Main Result

We start with some general notions of the theory of permutation groups and, in particular, some definitions and terminology. A standard reference is [12].

DEFINITION 1. Let π be an element of the symmetric group S_n and i a natural number not greater than n. We will say that π presents in i an anti-exceedance if $i\pi \leqslant i$; otherwise we say that π presents in i an exceedance.

EXAMPLE. In the permutation $(1\,8\,5)(2\,7)(3)(4\,9\,6\,10)$ of S_{10} the anti-exceedances are:

$$8\to 5;\qquad 5\to 1;\qquad 7\to 2;\qquad 3\to 3;\qquad 9\to 6;\qquad 10\to 4.$$

DEFINITION 2. If π is an element of the symmetric group S_n and i a natural number not greater than n we denote by $e_\pi(i)$ the difference $|i\pi - i|$.

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DEFINITION 3. The minimal degree of a permutation group G of degree n is the natural number $n-k$, where k is the maximal number of points fixed by the permutations of $G^* = G - \{id\}$.

The main result of this work is the following:

THEOREM 1. *Let G be a permutation group of degree n, minimal degree $n-k$ and containing the cycle $\alpha = (1\,2 \cdots n)$, and let $d \leq n-1$ be a natural number. If $n > 2kd - k$ then every permutation $\pi \in G - \langle \alpha \rangle$ has at least $d+1$ anti-exceedances.*

PROOF. Suppose that $\pi \in G - \langle \alpha \rangle$ has at most d anti-exceedances. Let $D = \{i \in [n]/i\pi \leq i\}$ be the set of the anti-exceedances and let $M = [n] - D$ be its complement, the set of the exceedances.

The fact that the minimal degree of G is $n-k$ means that two different permutations π and σ cannot act in the same way on $k+1$ different integers $i_1, \ldots, i_{k+1}$ because otherwise $\pi\sigma^{-1}$ fixes $i_1, \ldots, i_{k+1}$.

We also observe that the action of the permutation α^p consists of increasing the integers smaller than $n-p+1$ by a quantity p and decreasing the ones greater than $n-p$ by a quantity $n-p$. This implies that $k+1$ elements of M cannot have the same differences e_π: if $i_1, \ldots, i_{k+1} \in M$ and $e_\pi(i_1) = \cdots = e_\pi(i_{k+1}) = r$ the action of π coincides with the action of α^r and this has been excluded. In the same way we can show that the e_π's of $k+1$ elements of D cannot all be equal.

Moreover, if there are h elements of M for which $e_\pi = p$ (so that necessarily they must be smaller than $n-p$), then for no more than $k-h$ elements of D we have $e_\pi = n-p$, because the action of π cannot coincide with the action of α^p on more than k elements. Let $d_i = e_\pi(i)$, $i \in D$, $m_j = e_\pi(j)$, $j \in M$, and $c_i = n - d_i$.

The sum of the $e_\pi(i)$'s for $i \in D$ is equal to the sum of the $e_\pi(i)$'s with $i \in M$, as shown by the following chain of equalities:

$$\sum_{i=1}^{n} i = \sum_{i=1}^{n} i\pi; \qquad \sum_{i\in D} i + \sum_{i \in M} i = \sum_{i\in D} i\pi + \sum_{i\in M} i\pi;$$
$$\sum_{i\in D} i - \sum_{i\in D} i\pi = \sum_{i\in M} i\pi - \sum_{i\in M} i; \qquad \sum_{i\in D} (i - i\pi) = \sum_{i\in M} (i\pi - i).$$

This can be written as:

$$\sum_{i\in D} d_i = \sum_{j\in M} m_j; \qquad \sum_{i\in D} (n - c_i) = \sum_{j\in M} m_j;$$

and since we suppose that π has at most d anti-exceedances:

$$dn - \sum_{i\in D} c_i \geq \sum_{j\in M} m_j; \qquad dn \geq \sum_{i\in D} c_i + \sum_{j\in M} m_j.$$

Now on the right-hand side there are n natural numbers between 1 and n, and according to the above statements an integer m cannot appear more than k times among them. Therefore, if we divide n by k, $n = uk + v$ with $0 \leq v < k$, we have that the quantity on the right-hand side attains its minimum when the numbers $1, \ldots, u$

appear k times and the number $u+1$ appears v times. Thus:

$$dn \geqslant 1 \cdot k + 2 \cdot k + \cdots + u \cdot k + (u+1) \cdot v = (u+1)\left(\frac{uk}{2} + v\right)$$

$$= (u+1)\left(\frac{uk+v}{2} + \frac{v}{2}\right) = \left(\frac{n-v}{k} + 1\right)\left(\frac{n}{2} + \frac{v}{2}\right)$$

which leads to:

$$n^2 + (k - 2kd)n + (vk - v^2) \leqslant 0.$$

Since $vk - v^2 \geqslant 0$, the latter holds if

$$n^2 + (k - 2kd)n \leqslant 0;$$

that is,

$$n \leqslant 2kd - k.$$

This completes the proof. □

As a reference we present Table 1.

For all the permutation groups of degree between 3 and 7 that contain the cycle $\alpha = (1\,2 \cdots n)$, Table 1 gives the order of the group, the maximal number k of elements fixed by the non-identity permutations and the minimal number d of anti-exceedances of the elements of $G - \langle \alpha \rangle$. This table has been determined starting from the tables regarding permutation groups contained in the appendix of [10], and using the Cayley system (see [1] and [2]). In Table 1, D_n, A_n and S_n denote the dihedral, alternating and symmetric groups on n elements, respectively. For the groups of type $G_{k,n}$ the two indices correspond to the index and the order of the cyclic

TABLE 1

n	G	$o(G)$	k	d
3	S_3	6	1	2
4	D_4	8	2	2
	S_4	24	2	2
5	D_5	10	1	3
	$G_{4,5}$	20	1	3
	A_5	60	2	2
	S_5	120	3	2
6	D_6	12	2	3
	$G_{3,6}$	18	3	2
	$G_{4,6}$	24	4	2
	$G_{6,6}$	36	3	2
	$G_{8,6}$	48	4	2
	$G_{12,6}$	72	4	2
	$G_{20,6}$	120	2	2
	S_6	720	4	2
7	D_7	14	1	4
	$G_{3,7}$	21	1	4
	$G_{6,7}$	42	1	4
	$G_{24,7}$	168	3	3
	A_7	2520	4	2
	S_7	5040	5	2

subgroup $\langle \alpha \rangle$, respectively. Their generators are the following:

$$G_{4,5} = \langle (1\,2\,3\,4\,5), (1\,2\,4\,3) \rangle$$
$$G_{3,6} = \langle (1\,2\,3\,4\,5\,6), (2\,4\,6) \rangle$$
$$G_{4,6} = \langle (1\,2\,3\,4\,5\,6), (1\,2\,3)(4\,5\,6) \rangle$$
$$G_{6,6} = \langle (1\,2\,3\,4\,5\,6), (3\,5)(4\,6) \rangle$$
$$G_{8,6} = \langle (1\,2\,3\,4\,5\,6), (1\,4)(2\,3\,5\,6) \rangle$$
$$G_{12,6} = \langle (1\,2\,3\,4\,5\,6), (2\,4) \rangle$$
$$G_{20,6} = \langle (1\,2\,3\,4\,5\,6), (2\,4\,5\,6) \rangle$$
$$G_{3,7} = \langle (1\,2\,3\,4\,5\,6\,7), (2\,3\,5)(4\,7\,6) \rangle$$
$$G_{6,7} = \langle (1\,2\,3\,4\,5\,6\,7), (2\,4\,3\,7\,5\,6) \rangle$$
$$G_{24,7} = \langle (1\,2\,3\,4\,5\,6\,7), (2\,4\,3)(1\,6\,5) \rangle$$

Remark. Some of the groups above are better known under other names or presentations. For example, the group $G_{24,7}$ is hisomorphic to $L_2(7)$, the simple group of order 168.

3. Anti-exceedances in Frobenius Groups

Definition 4. A transitive permutation group G is a Frobenius group if it has degree n and minimal degree $n-1$.

A Frobenius group G is a transitive permutation group in which only the identity fixes two points. Moreover, the permutations of G that fix no element form together with the identity a normal subgroup N (called the Frobenius kernel), the order of which is equal to the degree n. The subgroup N is regular; that is, all its permutations can be decomposed into disjoint cycles having the same length. Outside the kernel are all the elements fixing a point, divided into the various stabilizers all conjugate to one another and with trivial intersection, as schematized in Figure 1.

If H_i denotes the stabilizer of i we have $|H_i| = [G: N]$ and G is the semidirect product of H_i and N; the H_i's are called Frobenius complements (of the kernel N). In a Frobenius group containing the cycle $\alpha = (1\,2 \cdots n)$, the kernel is just the cyclic subgroup generated by α. References for Frobenius groups are [5], [7], [11] and [12].

In the case of a Frobenius group, as we will see, it is possible to give a much more precise answer to the problem of anti-exceedances than that of Theorem 1.

Let us consider, for example, the dihedral group on n elements, n odd. These are all Frobenius groups. In this case $G - \langle \alpha \rangle$ is a set of n permutations which are a product of $(n-1)/2$ transpositions; each of them fixes an element and exchanges all the other

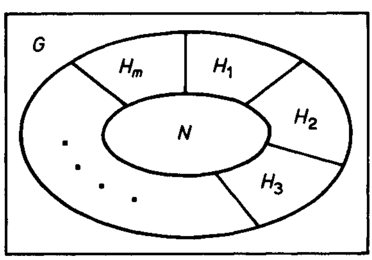

Figure 1

ones in pairs. All these permutations present exactly $(n+1)/2$ anti-exceedances, one for each transposition plus the one given by the fixed point. The following theorem says that it is possible to generalize this fact to any Frobenius group.

THEOREM 2. *Let G be a Frobenius group of degree n which contains the cycle $\alpha = (1\,2 \cdots n)$. Then any permutation $\pi \in G - \langle \alpha \rangle$ has exactly $(n+1)/2$ anti-exceedances.*

PROOF. Since G is a Frobenius group, any two permutations π and σ cannot act in the same way on two different elements i_1 and i_2 because in that case $\pi\sigma^{-1}$ would fix i_1 and i_2.

Let π be any permutation of G. Let D and M be the sets of the anti-exceedances and of the exceedances, respectively, and let $d = \mathrm{card}(D)$. By the same arguments as used in Theorem 1 we can prove that the elements of M all have distinct e_π, and also that the e_π's of the elements of D are all distinct.

Moreover, if there exists an element i in M having a difference equal to p (it must necessarily be $i \leqslant n-p$), then this implies that no element in D has e_π equal to $n-p$.

As in Theorem 1, let us denote $d_i = e_\pi(i)$ for each $i \in D$ and $m_j = e_\pi(j)$ for each $j \in M$. The d_i's are all different and so are the m_j's; let $c_i = n - d_i$; the c_i's are also all distinct and they are different from the m_j's.

As in Theorem 1 we have:

$$\sum_{i\in D} d_i = \sum_{j\in M} m_j; \quad \sum_{i\in D} (n-c_i) = \sum_{j\in M} m_j; \quad dn - \sum_{i\in D} c_i = \sum_{j\in M} m_j; \quad dn = \sum_{i\in D} c_i + \sum_{j\in M} m_j.$$

Now on the right-hand side there are n natural numbers between 1 and n, all different, so their sum is $n(n+1)/2$; hence we obtain:

$$dn = n(n+1)/2.$$

This completes the proof. □

COROLLARY 1. *The bound for n in Theorem 1 is the best possible: that is, there exist $k \in \mathbb{N}$ and a group G of minimal degree k, of degree $2kd - k$ and containing the cycle $(1\,2 \cdots n)$ in which there is a permutation that has no more than d anti-exceedances.*

PROOF. It suffices to take $k = 1$ and consider a Frobenius group G of degree $2d - 1$. By the previous theorem, every permutation of G has exactly d anti-exceedances. □

However, this raises the problem of the optimality of the bound for other values of k.

COROLLARY 2. *If G is a Frobenius group of degree n and cyclic Frobenius kernel then n is odd.*

PROOF. The simple combinatorial proof consists of observing that in this case the kernel is exactly the cyclic subgroup generated by α and so (hypotheses are the same as those in the theorem) we have $n = 2d - 1$; this can be extended to groups the kernel of which is generated by cycles of length n and that are not powers of α, since these groups can be obtained by conjugation from the ones we have considered (in other words, by renaming the elements on which the permutations act). □

REMARK. The algebraic reason of this fact is the following: if n is even, the element $\alpha^{n/2}$ is the only involution of the kernel; if we conjugate it with an element

$\pi \in G - \langle \alpha \rangle$ we obtain $\pi\alpha^{n/2}\pi^{-1}$ which is still an involution and is still in $\langle \alpha \rangle$, because $\langle \alpha \rangle$ is normal; since $\alpha^{n/2}$ is the unique involution we have:

$$\pi\alpha^{n/2}\pi^{-1} = \alpha^{n/2}; \qquad \pi\alpha^{n/2} = \alpha^{n/2}\pi.$$

Now, a permutation fixing only one element cannot commute with one that fixes no elements. If we suppose that i is the point fixed by π we would have:

$$(i\alpha^{n/2})\pi = i\alpha^{n/2}\pi = i\pi\alpha^{n/2} = i\alpha^{n/2};$$

that is, π fixes $i\alpha^{n/2}$ and thus $i\alpha^{n/2} = i$, a contradiction.

EXAMPLES. We give a few examples of Frobenius groups containing the cycle $(1\,2\cdots n)$ that are not dihedral groups on n elements; from Table 1 we have the following groups:

$n = 5$: $G_{4;5} = \langle (1\,2\,3\,4\,5), (1\,2\,4\,3)\rangle$ of order 20 and $d = 3 = (5+1)/2$
$n = 7$: $G_{3;7} = \langle (1\,2\,3\,4\,5\,6\,7), (2\,3\,5)(4\,7\,6)\rangle$ of order 21
$G_{6;7} = \langle (1\,2\,3\,4\,5\,6\,7), (2\,4\,3\,7\,5\,6)\rangle$ of order 42
and for both of these groups $d = 4 = (7+1)/2$.

The first example of a Frobenius group containing the cycle $(1\,2\cdots n)$ and not equal to the dihedral group on n elements with n not a prime number can be found in S_{25}; it is the group of order 100 generated by $x = (1\,2\,3\cdots 2\,5)$ and $y = (1)(2\,8\,25\,19)(3\,15\,24\,12)(4\,22\,23\,5)(6\,11\,21\,16)(7\,18\,20\,9)(10\,14\,17\,13)$, the elements of which in $G - \langle x \rangle$ all have $(25+1)/2 = 13$ anti-exceedances.

When the Frobenius kernel does not contain the cycle $(1\,2\cdots n)$ or when it is not cyclic, no answer is given by the latter theorem. It is not difficult to see that we obtain by conjugation some groups the permutations of which have an arbitrary number of anti-exceedances. So the problem seems to be intractable. A partial answer is given in the next theorem. First we give a remark.

REMARK. If τ is a cycle $(i_1\cdots i_m)$, $m > 1$, and τ presents t anti-exceedances, then the inverse cycle presents $m - t$ anti-exceedances (τ^{-1} presents an anti-exceedance iff τ presents an exceedance).

NOTATION. If G is a permutation group and $\pi \in G$ is a permutation, we will denote by d_π the number of anti-exceedances of the permutation π, by $D_G = \sum_{\pi\in G} d_\pi$ the sum of the numbers of anti-exceedances of all the permutations of G, and by $D_G^* = \sum_{\pi\in G^*} d_\pi$ the sum of the numbers of anti-exceedances of all the permutations of G not equal to the identity.

THEOREM 3. *Let G be a Frobenius group of degree n, of kernel N of index h, and of complement H. Then*

$$D_G = |G|\frac{n+1}{2}$$

and, in particular,

$$D_N = n\frac{n+1}{2} = |N|\frac{n+1}{2} \qquad D_H^* = (h-1)\frac{n+1}{2} = (|H|-1)\frac{n+1}{2}.$$

In other words, the mean of the number of anti-exceedances in the whole group G, in its kernel and in the complements without identity is always $(n+1)/2$.

PROOF. The $n-1$ permutations of N^* fix no element, so they are products of cycles of length greater than 1. If $\pi \in N$ and $\pi \neq \pi^{-1}$ (that is, if π is not an involution), we have $d_\pi + d_{\pi^{-1}} = n$; suppose that π is an involution so it is a product of transpositions and, more precisely, since it must move all the elements, it is a product of $n/2$ transpositions and hence it presents $n/2$ anti-exceedances. If i_N denotes the number of involutions in N, then $(n-1-i_N)/2$ is the number of pairs of elements (π, π^{-1}) with $\pi \in N$ and $\pi \neq \pi^{-1}$. We have:

$$D_N^* = \frac{n-1-i_N}{2} n + i_N \frac{n}{2} = \frac{n-1}{2} n$$

If we add the n anti-exceedances of the identity we obtain:

$$D_N = \frac{n-1}{2} n + n = n \frac{n+1}{2}.$$

If H is a complement of N, any permutation of $H - \{id\}$ fixes exactly one element. Hence in its decomposition into disjoint cycles there appears exactly one cycle of length one, and this cycle will appear also in its inverse, thus:

$$\forall \pi \in H - \{id\},\ d_\pi + d_{\pi^{-1}} = n + 1.$$

If H contains an involution, it must be a product of $(n-1)/2$ transpositions, since only one integer can be fixed; thus it presents

$$\frac{n-1}{2} + 1 = \frac{n+1}{2}$$

anti-exceedances. If i_H denotes the number of involutions of H, then $(h-1-i_H)/2$ is the number of pairs of elements (π, π^{-1}) with $\pi \in H$ and $\pi \neq \pi^{-1}$ and we have:

$$D_H^* = \frac{h-1-i_H}{2}(n+1) + i_N \frac{n+1}{2} = (h-1)\frac{n+1}{2}.$$

Since the n conjugates of H have trivial intersection with H, it suffices to multiply this quantity by n to obtain the total number of anti-exceedances outside N; adding D_N to this number we obtain:

$$D_G = n\frac{n+1}{2} + n(h-1)\frac{n+1}{2} = nh\frac{n+1}{2} = |G|\frac{n+1}{2}. \qquad \square$$

COROLLARY 3. *There does not exist a Frobenius groups the kernel and complement of which have even order.*

PROOF. This follows directly from the third identity given in the previous theorem:

$$D_H^* = (h-1)\frac{n+1}{2}.$$

When n is even, the right-hand side of this expression has an integer value only if h is odd. $\square$

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